



INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

GAUSSIAN INTEGER SOLUTIONS FOR THE ELLIPTIC PARABOLOID

$$x^2 + 2y^2 = 4z$$

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ABSTRACT

The elliptic paraboloid represented by the ternary quadratic diophantine equation $x^2 + 2y^2 = 4z$ is considered and analyzed for its Gaussian integer solutions. Also, knowing a solution, general formulas for generating sequence of Gaussian integer solutions for the above equation are presented.

KEYWORDS: Elliptic paraboloid, Ternary quadratic, Gaussian integer.

INTRODUCTION

The ternary quadratic diophantine equations are rich in variety and have been attracted by many Mathematicians [1-23] and the solutions are represented by non-zero distinct integers. In this context, note that an extension of ordinary integers into complex numbers is the Gaussian integers. In [24] Gaussian integer solutions of the pythagorean equation have been presented. The Gaussian integer solutions for the special equations $z^2 = y^2 + Dx^2$ and $x^2 + y^2 = 2z^2$ are presented in [25,26] respectively. Also, in [27] the elliptic paraboloid equation $3x^2 + 2y^2 = 3z$ has been analyzed for Gaussian integer solutions. This has motivated us to search for Gaussian integer solutions of other ternary quadratic diophantine equations. This communication concerns with the problem of obtaining Gaussian integer solutions of the elliptic paraboloid $x^2 + 2y^2 = 4z$. Also, knowing a solution, general formulas for generating sequence of Gaussian integer solutions for the above equation are presented.

METHOD OF ANALYSIS

Consider the ternary quadratic equation representing the elliptic paraboloid given by

$$x^2 + 2y^2 = 4z \tag{1}$$

We present below different patterns of Gaussian integer solutions to (1)

Pattern 1:

Introducing the linear transformations

$$x = 2f + i2, \quad y = -1 + if, \quad z = e + if \tag{2}$$

in (1), it is written as

$$2f^2 - 2e - 1 = 0$$

which is satisfied by

$$f = 2k + 1, \quad e = (2k^2 + 2k)$$

Substituting the values of e & f in (2), the Gaussian integer solutions to (1) are given by

$$\begin{aligned} x &= (4k + 2) + i2 \\ y &= -(1 + i(2k + 1)) \\ z &= 2k(k + 1) + i(2k + 1) \end{aligned}$$

Pattern 2:

Introducing the transformations

$$x = 2f + 2i, \quad y = -1 + if, \quad z = e^2 + if \tag{3}$$

in (1), it is written as

$$f^2 = 2e^2 + 1$$

which is a Pell equation, whose initial solution is

$$e_0 = 2, \quad f_0 = 3$$

The general solution of equation (1) is given by

$$f_n = \frac{1}{2}F_n$$

$$e_n = \frac{1}{2\sqrt{2}}E_n$$

Where $F_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}$

$$E_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} \quad n=-1,0,\dots\dots$$

Substituting the values of E_n & F_n in (3), the Gaussian integer solutions to (1) are given by

$$x_n = \left[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right] + i2$$

$$y_n = -1 + \frac{i}{2} \left[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right]$$

$$z_n = \frac{1}{2\sqrt{2}} \left[(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} \right]^2 + \frac{i}{2} \left[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right]$$

Pattern 3:

Consider

$$\left. \begin{aligned} x &= A + i2B \\ y &= A + iD \\ z &= (B^2 + 2D^2) + iA(B + D) \end{aligned} \right\} \tag{4}$$

Using (4) in (1), it is written as

$$3A^2 - 8B^2 = 10D^2 \tag{5}$$

Introducing the linear transformations

$$\left. \begin{aligned} A &= X + 8T \\ B &= X + 3T \end{aligned} \right\} \tag{6}$$

in (5), we have

$$X^2 + 2D^2 = 24T^2 \tag{7}$$

Assume $T = a^2 + 2b^2$ and $24 = (4 + i2\sqrt{2})(4 - i2\sqrt{2})$

Using factorization method we have

$$X = 4a^2 - 8b^2 - 8ab$$

$$D = 2a^2 - 4b^2 + 8ab$$

substituting the values of X, T, D in (6) and (4), we have

$$x = (12a^2 + 8b^2 - 8ab) + i2(7a^2 - 2b^2 - 8ab)$$

$$y = (12a^2 + 8b^2 - 8ab) + i(2a^2 - 4b^2 + 8ab)$$

$$z = \{(7a^2 - 2b^2 - 8ab)^2 + 2(2a^2 - 4b^2 + 8ab)^2\} + i\{(12a^2 + 8b^2 - 8ab)(9a^2 - 6b^2)\}$$

which represents the Gaussian integer solutions of (1).

Pattern 4:

Consider

$$\left. \begin{aligned} x &= 2A + i2B \\ y &= A + i2D \\ z &= (2B^2 + D^2) + i2A(B + D) \end{aligned} \right\} \tag{8}$$

Using (8) in (1), it is written as

$$A^2 - 2B^2 = 2D^2 \tag{9}$$

Introducing the linear transformations

$$\left. \begin{aligned} A &= 2X + 2T \\ B &= 2X + T \end{aligned} \right\} \quad (10)$$

in (9), we have

$$T^2 = 2X^2 + D^2 \quad (11)$$

which is satisfied by

$$\begin{aligned} T &= 2rs \\ D &= 10r^2 - s^2 \\ X &= 10r^2 + s^2 \end{aligned}$$

Substituting the values of X, T, D in (10) and (8), we have

$$\begin{aligned} x &= 2[(4r^2 + 2s^2 + 4rs) + i(2r^2 + s^2 + 4rs)] \\ y &= 2[(2r^2 + s^2 + 2rs) + i(2r^2 - s^2)] \\ z &= \{2(2r^2 + s^2 + 4rs)^2 + (2r^2 - s^2)^2\} \\ &\quad + i2\{(4r^2 + 2s^2 + 4rs)(2r^2 + 2rs)\} \end{aligned}$$

which represents the Gaussian integer solutions of (1).

Pattern 5:

Note that (11) can be represented as the system of double equations in different ways as follows.

System 1:

$$T + D = 2X, \quad T - D = X$$

System 2:

$$T + D = X^2, \quad T - D = 2$$

solving each of the above two systems, we obtain the values of X, T, D

Substituting the above values of X, T, D in (10) and (8), the corresponding Gaussian integer solutions to (1) are obtained and they are represented as follows.

Solutions for system 1:

$$\begin{aligned} x &= 2k(10 + i7) \\ y &= 2k(5 + i) \\ z &= 5k^2(3 + i32) \end{aligned}$$

Solutions for system 2:

$$\begin{aligned} x &= (8k^2 + 8k + 4) + i(4k^2 + 8k + 2) \\ y &= (4k^2 + 4k + 2) + i(4k^2 - 2) \\ z &= \{2(2k^2 + 4k + 1)^2 + (2k^2 - 1)^2\} \\ &\quad + i\{16(2k^2 + 2k + 1)(k^2 + k)\} \end{aligned}$$

Pattern 6:

Equation (9) can be rewritten as

$$A^2 = 2(B^2 + D^2) \quad (12)$$

$$\text{Assume } A = \alpha^2 + \beta^2 \text{ and } 2 = (1 + i)(1 - i) \quad (13)$$

Substituting (13) in (12) and using factorization method we have

$$\begin{aligned} B &= \frac{\alpha^2 - \beta^2 + 2\alpha\beta}{2} \\ D &= \frac{\beta^2 - \alpha^2 + 2\alpha\beta}{2} \end{aligned}$$

Case (i):

$$\text{Let } \alpha = 2p, \quad \beta = 2q$$

Case (ii):

$$\text{Let } \alpha = (2p+1), \quad \beta = (2q + 1)$$

Solutions for case(i) :

$$\begin{aligned} x &= 8(p^2 + q^2) + i[4(p^2 - q^2) + 8pq] \\ y &= 4(p^2 + q^2) + i[8pq - 4(p^2 - q^2)] \end{aligned}$$

$$z = 2\{16p^2q^2 + 4(p^2 - q^2)^2 + 16pq(p^2 - q^2)\} + i8(p^2 + q^2)(8pq)$$

Solutions for case(ii) :

$$x = 2\{[4(p^2+q^2) + 4(p+q) + 2] + i[(4pq+2(p+q)+1)+2(p^2-q^2+p-q)]\}$$

$$y = (4p^2 + 4q^2 + 4p + 4q + 2) + i2(2q^2 - 2p^2 + 4pq + 2q + 2)$$

$$Z = \{2[(2p+1)(2q+1) + 2(p^2 - q^2 + p - q)]^2 + [(2p+1)(2q+1) - 2(p^2 - q^2 + p - q)]^2\} + i2\{4(p^2 + q^2) + 4(p+q) + 2\}(8pq + 4(p+q) + 2)\}$$

Pattern 7:

$$\text{Let } B = E + F, \quad D = E - F$$

in (9) can be written as

$$A^2 = (2E)^2 + (2F)^2 \tag{14}$$

Which is satisfied by

$$2E = p^2 - q^2$$

$$2F = 2pq$$

$$A = p^2 + q^2 \quad \text{Provided } p > q > 0$$

Replace p by 2p and q by 2q we have,

$$B = 2(p^2 - q^2) + 4pq$$

$$D = 2(p^2 - q^2) - 4pq$$

Substituting the value of B, D, A in (8), we have the Gaussian integer solution are given by

$$x = 2\{(p^2 + q^2) + i(2(p^2 - q^2) + 4pq)\}$$

$$y = (p^2 + q^2) + i4(p^2 - q^2 - 2pq)$$

$$z = \{2[2(p^2 - q^2) + 4pq]^2 + [2(p^2 - q^2) - 4pq]^2\} + i8[(p^2 + q^2)(p^2 - q^2)]$$

GENERATION OF SOLUTIONS

Now, given Gaussian integer solution to (1), can an infinite number of Gaussian integer solutions be generated? Albeit tacitly, the answer for this question is in the affirmative.

Generation 1:

Let (x_0, y_0, z_0) be a Gaussian integer solution of (1).

$$\text{Let } x_1 = x_0 + h, \quad y_1 = y_0 + h, \quad z_1 = z_0 + h^2 \tag{15}$$

be the second Gaussian integer solutions to (1). Here 'k' is any given non-zero integer and 'h' is a non-zero Gaussian integer.

Substituting (15) in (1) and simplifying, we have

$$h = 2[x_0 + 2y_0]$$

Using this value of 'h' in (15), we have

$$x_1 = 3x_0 + 4y_0, \quad y_1 = 2x_0 + 5y_0, \quad z_1 = z_0 + (2x_0 + 4y_0)^2$$

Repeating the above process, the general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = \frac{1}{6}[2(7^n + 2)x_0 + 4(7^n - 1)y_0]$$

$$y_n = \frac{1}{6}\{2(7^n - 1)x_0 + 2(2 \cdot 7^n + 1)y_0\}$$

$$z_n = z_0 + (2x_0 + 4y_0)^2 \left[\frac{(7^{2n} - 1)}{48} \right]$$

Generation 2:

Let (x_0, y_0, z_0) be a Gaussian integer solution of (1).

The second solution is obtained by setting

$$x_1 = h - 3x_0, \quad y_1 = h - 3y_0, \quad z_1 = 9z_0$$

where 'h' is a non-zero Gaussian integer given by

$$h = 2[x_0 + 2y_0]$$

The repetition of the above process leads to the general form of Gaussian integer solution to (1) given by

$$x_n = \frac{1}{3} \{ [3^n + 2(-3)^n]x_0 + 2[3^n - (-3)^n]y_0 \}$$

$$y_n = \frac{1}{3} [3^n - (-3)^n]x_0 + [2(3)^n + (-3)^n]y_0$$

$$z_n = 9^n z_0$$

CONCLUSION

In this paper, we have presented infinitely many Gaussian integer solutions to the elliptic paraboloid given by $x^2 + 2y^2 = 4z$. Also, we have illustrated a method for generating sequence of Gaussian integer solutions for the given equation, knowing its initial solution. As the ternary quadratic equation are rich in variety, one may search for Gaussian integer solutions for other choices of elliptic paraboloids.

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